

### Abstract

Any topological group  $G$  acts on its space  $S(G)$  of closed subgroups by conjugation. We study the simplex of conjugation-invariant probability measures on  $S(G)$  in the special case in which  $G$  is one of the lamplighter groups  $(\mathbb{Z}/p\mathbb{Z})^n \wr \mathbb{Z}$ . The main result shows this simplex has a canonical Poulsen subsimplex whose complement has only a countable number of extreme points.

# Invariant random subgroups of the lamplighter group

Lewis Bowen<sup>\*</sup>

Texas A&M University

Rostislav Grigorchuk<sup>†</sup>

Texas A&M University

Rostyslav Kravchenko<sup>‡</sup>

University of Chicago

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## 1 Introduction

Let  $G$  be a discrete countable group and  $S(G)$  be the set of all subgroups of  $G$  equipped with the topology induced by the Tychonoff topology of the product space  $\{0, 1\}^G$  (a subgroup  $H$  is identified with its characteristic function  $1_H \in \{0, 1\}^G$ ). The group  $G$  acts on  $S(G)$  continuously by conjugation  $g \cdot H := gHg^{-1}$ . Let  $M(G)$  denote the space of all conjugation-invariant Borel probability measures on  $S(G)$  supplied with the weak\* topology. A random subgroup with law in  $M(G)$  is called an IRS (invariant random subgroup). See [AGV12, Bo12] for more references to this subject. A problem of general interest is: “for each ‘interesting’ group, describe the simplex of continuous invariant measures on  $S(G)$ ”. Of course “to describe” must be specified and depends on each particular case.

Being a totally disconnected compact metrizable space,  $S(G)$  is completely characterized by its Cantor-Bendixon rank and existence (or absence) of perfect kernel  $\mathcal{K}$  (which is homeomorphic to a Cantor set). Since  $\mathcal{K}$  is invariant, it is mostly interesting to focus on those IRS’s supported on  $\mathcal{K}$ . For  $\mathcal{K}$  to be nonempty the group  $G$  must have uncountably many subgroups which is not always the case (for instance, f.g. nilpotent and polycyclic groups have only countably many subgroups). The Lamplighter groups  $\mathcal{L}_n = (\mathbb{Z}/p\mathbb{Z})^n \wr \mathbb{Z}$  (for fixed prime  $p$ ) are the simplest examples of metabelian groups with uncountably many subgroups.

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<sup>\*</sup>email:lpbowen@math.tamu.edu

<sup>†</sup>email:grigorch@math.tamu.edu

<sup>‡</sup>email:rkchenko@gmail.com

In [Bo12] among other things Bowen showed that there is a “zoo” of IRS’s of the free group  $F_m$  of rank  $m \geq 2$  and that the “essential part” of the simplex  $M(F_m)$  is a Poulsen simplex; which means that its extreme points are dense. In this note we show that a similar result holds already for  $\mathcal{L}_n$  (see Theorem 3.3).

## 2 The space of subgroups of $\mathcal{L}_n$

Fix a prime number  $p$ . We consider groups

$$\mathcal{L}_n = (\mathbb{Z}/p\mathbb{Z})^n \wr \mathbb{Z} = \oplus_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})^n \rtimes \mathbb{Z}.$$

Let  $\mathcal{A}_n$  denote the subgroup  $\oplus_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})^n$  of  $\mathcal{L}_n$ . Let  $x$  denote the right shift by one on  $\mathcal{A}_n$ , considered as a generator of the active group  $\mathbb{Z}$ . Elements of  $\mathcal{L}_n$  are written as  $(v, s)$  where  $v \in \mathcal{A}_n$  and  $s \in \mathbb{Z}$ . If  $g = (v, s)$  and  $h = (w, t)$  are elements of  $\mathcal{L}_n$  then direct computation gives that  $gh = (v + x^s w, s + t)$  and  $g^{-1} = (-x^{-s} v, -s)$ .

Let us introduce the notation  $\varphi_t(x) = 1 + x + \dots + x^{t-1}$ . The next result is [GK12, Lemma 3.1].

**Lemma 2.1.** *Let  $V$  be a subgroup of  $\mathcal{L}_n$ . Then it defines a triple  $(s, V_0, v)$ , where  $s \in \mathbb{N}$  is such that  $s\mathbb{Z}$  is the image of projection of  $V$  on  $\mathbb{Z}$ ,  $V_0 = V \cap \mathcal{A}_n$ , satisfies  $x^s V_0 = V_0$ , and  $v \in \mathcal{A}_n$  is such that  $(v, s) \in V$ . The element  $v$  is uniquely determined up to addition of elements from  $V_0$ . For  $s = 0$  one can choose  $v = 0$ .*

*Conversely any triple  $(s, V_0, v)$  with such properties gives rise to a subgroup of  $\mathcal{L}_n$ . Two triples  $(s, V_0, v)$  and  $(s', V'_0, v')$  define the same subgroup if and only if  $s = s'$ ,  $V_0 = V'_0$  and  $v + V_0 = v' + V_0$ .*

*Moreover,  $V \subset V'$  if and only if  $s'|s$ ,  $V_0 \subset V'_0$  and  $v = \varphi_{s/s'}(x^{s'})v' \pmod{V'_0}$ .*

Define  $\pi_1 : S(\mathcal{L}_n) \rightarrow \mathbb{N}$  by  $\pi_1(V) = s$  where  $s\mathbb{Z}$  is the image of projection of  $V$  on  $\mathbb{Z}$ . Define  $\pi_2 : S(\mathcal{L}_n) \rightarrow S(\mathcal{A}_n)$  by  $\pi_2(V) = V \cap \mathcal{A}_n$ .

Let  $R = \mathbb{F}_p[x, x^{-1}]$ , and define the structure of an  $R$ -module on  $\mathcal{A}_n$  by setting  $x\omega$  equal the shift of  $\omega$  by  $+1$ .

**Lemma 2.2.** *There are only a countable number of subgroups  $V$  of  $\mathcal{L}_n$  with  $\pi_1(V) > 0$ .*

*Proof.* Let  $V$  be such a subgroup,  $s = \pi_1(V)$ ,  $V_0 = V \cap \mathcal{A}_n$ , and  $v \in \mathcal{A}_n$  be such that  $(v, s) \in V$ . Define a new  $R$ -module structure  $N$  on  $\mathcal{A}_n$  by  $x * \omega = x^s \omega$ . Then  $V_0$  is an  $R$ -submodule of  $N$ . Note that since  $s > 0$ ,  $N$  is isomorphic to  $R^{ns}$  as an  $R$ -module (if  $e_i$ ,  $i \in [1, n]$  is the basis of  $\mathcal{A}_n$  then the basis of  $N$  is  $x^j e_i$ ,  $i \in [1, n]$  and  $j \in [0, s-1]$ ).

Note that there is only a countable number of submodules of  $R^k$ . Indeed, since  $R$  is a PID and  $R^k$  is finitely generated, any submodule is also finitely generated. Since  $R$  is countable,  $R^k$  is also countable, hence there is only a countable number of finite subsets of  $R^k$ .

Since  $\mathcal{A}_n$  is countable, we have that for each  $s > 0$  there is only a countable number of possible triples  $(s, V_0, v)$ . □

**Lemma 2.3.** *The Cantor-Bendixon rank of  $S(\mathcal{L}_n)$  is 1. The perfect kernel of  $S(\mathcal{L}_n)$  is  $S(\mathcal{A}_n)$ .*

*Proof.* It suffices to show that  $S(\mathcal{A}_n)$  does not have isolated points. Let  $H$  be a subgroup of  $\mathcal{A}_n$ . There is a countable basis  $f_i$  of  $\mathcal{A}_n$ , indexed by a set  $I$ , and a partition  $I = I_1 \amalg I_2$  such that  $\{f_i : i \in I_1\}$  is a basis of  $H$ . At least one of the sets  $I_1$  or  $I_2$  is infinite. If  $I_q$  is infinite, then the collection of subgroups  $H_j$ ,  $j \in I_q$ , with  $H_j$  being the subgroup generated by  $\{f_i | i \in I_1 \Delta \{j\}\}$ , does not contain  $H$  but has  $H$  as an accumulation point.  $\square$

**Lemma 2.4.** *The map  $\pi_2$  is continuous. The map  $\pi_1$  is Borel and conjugation-invariant in the sense that  $\pi_1(gVg^{-1}) = \pi_1(V)$  for any  $g \in \mathcal{L}_n, V \in S(\mathcal{L}_n)$ .*

*Proof.* This is an exercise.  $\square$

### 3 Conjugation-invariant measures

For any group  $G$ , let  $M^e(G)$  denote the space of ergodic measures of  $M(G)$ . Let  $M_s(\mathcal{L}_n) \subset M^e(\mathcal{L}_n)$  be those ergodic measures  $\mu$  with  $\mu(\pi_1^{-1}(s)) = 1$ . By Lemma 2.4,  $M^e(\mathcal{L}_n)$  is the disjoint union of  $M_s(\mathcal{L}_n)$  over  $s \in \{0, 1, 2, 3, \dots\}$ .

**Lemma 3.1.** *If  $s > 0$  and  $\mu \in M_s(\mathcal{L}_n)$  then  $\mu$  is supported on a single finite conjugacy class. Therefore  $\cup_{s=1}^{\infty} M_s(\mathcal{L}_n)$  is countable.*

*Proof.* By Lemma 2.2,  $\mu$  is supported on a countable set. Because  $\mu$  is ergodic, this implies it is supported on single conjugacy class  $\mathcal{C} \subset S(\mathcal{L}_n)$ . Therefore it is the uniform measure on  $\mathcal{C}$  which implies that  $\mathcal{C}$  is finite.  $\square$

Let  $M_x(\mathcal{A}_n)$  be the space of shift-invariant Borel probability measures on  $S(\mathcal{A}_n)$ .

**Lemma 3.2.** *The map  $\pi_2$  pushes forward to an affine map  $(\pi_2)_* : M(\mathcal{L}_n) \rightarrow M_x(\mathcal{A}_n)$ . The restriction of  $(\pi_2)_*$  to  $M_0(\mathcal{L}_n)$  is an isomorphism.*

*Proof.* This is an exercise.  $\square$

Recall that a convex closed metrizable subset  $K$  of a locally convex linear space is a *simplex* if each point in  $K$  is the barycenter of a unique probability measure supported on the subset  $\partial_e K$  of extreme points of  $K$ . In this case,  $K$  is called a *Poulsen simplex* if  $\partial_e K$  is dense in  $K$ . It is called a *Bauer simplex* if  $\partial_e K$  is closed.

For example, an old result states that the space of all shift-invariant Borel probability measures on  $\{0, 1\}^{\mathbb{Z}}$  is a Poulsen simplex (we will not need this fact). It is known from [LOS78] that there is a unique Poulsen simplex up to affine isomorphism. Moreover, its set of extreme points is homeomorphic to the Hilbert space  $l^2$ . On the other hand, there are uncountably many non-isomorphic Bauer simplices. For example, let  $X$  be any compact metrizable space. Then the space  $P(X)$  of all Borel probability measures on  $X$  is a Bauer simplex with  $\partial_e P(X)$  homeomorphic to  $X$ .

**Theorem 3.3.**  $M_x(\mathcal{A}_n)$  is a Poulsen simplex. Therefore,  $M_0(\mathcal{L}_n)$  is also a Poulsen simplex.

*Proof.* Given an element  $w \in \mathcal{A}_n$ , we let  $w = (\dots, w_{-1}, w_0, w_1, w_2, \dots)$  with each  $w_k \in (\mathbb{Z}/p\mathbb{Z})^n$ . For  $i \leq j$  let  $\mathcal{X}^{[i,j]}$  be the subgroup of  $\mathcal{A}_n$  which consists of all  $w$  with  $w_k = 0$  if  $k \notin [i, j]$ . For  $i \leq j$ , let  $P^{i,j} : S(\mathcal{A}_n) \rightarrow S(\mathcal{X}^{[i,j]})$  be the intersection map  $P^{i,j}(H) = H \cap \mathcal{X}^{[i,j]}$ . It is easy to see that  $P^{i,j}$  is continuous. It induces a map  $P_*^{i,j} : M(\mathcal{A}_n) \rightarrow M(\mathcal{X}^{[i,j]})$ .

Let  $\mu \in M_x(\mathcal{A}_n)$ . It suffices to show that  $\mu$  is a limit point of ergodic measures in  $M_x(\mathcal{A}_n)$ . Let  $m > 0$  be an integer. Then  $P_*^{0,m-1}\mu$  is a measure on  $S(\mathcal{X}^{[0,m-1]})$ . So the infinite direct product  $(P_*^{0,m-1}\mu)^\mathbb{Z}$  is a measure on  $S(\mathcal{X}^{[0,m-1]})^\mathbb{Z}$ . Note that this measure is ergodic under the shift because it is Bernoulli.

Let  $\Phi : S(\mathcal{X}^{[0,m-1]})^\mathbb{Z} \rightarrow S(\mathcal{A}_n)$  be the map

$$\Phi(\dots, H_{-1}, H_0, H_1, \dots) = \bigoplus_{k \in \mathbb{Z}} x^{km} H_k$$

where each  $H_k \in S(\mathcal{X}^{[0,m-1]}) \subset S(\mathcal{A}_n)$ .

This map intertwines the shift on  $S(\mathcal{X}^{[0,m-1]})^\mathbb{Z}$  with the  $m$ -th power of the shift on  $S(\mathcal{A}_n)$ . So  $\Phi_*((P_*^{0,m-1}\mu)^\mathbb{Z})$  is invariant and ergodic under  $x^m$ . Finally, let

$$\mu_m := \frac{1}{m} \sum_{k=0}^{m-1} x^k \Phi_*((P_*^{0,m-1}\mu)^\mathbb{Z}).$$

This is a shift-invariant ergodic measure in  $M_x(\mathcal{A}_n)$ . It is ergodic because if  $E \subset S(\mathcal{A}_n)$  is any measurable shift-invariant set then for each  $k$ ,

$$x^k \Phi_*((P_*^{0,m-1}\mu)^\mathbb{Z})(E) \in \{0, 1\}$$

by ergodicity of  $\Phi_*((P_*^{0,m-1}\mu)^\mathbb{Z})$ . By shift invariance,  $x^k \Phi_*((P_*^{0,m-1}\mu)^\mathbb{Z})(E) = \Phi_*((P_*^{0,m-1}\mu)^\mathbb{Z})(E)$  for every  $k$ . So  $\mu_m(E) \in \{0, 1\}$  as required.

It now suffices to show that  $\lim_{m \rightarrow \infty} \mu_m = \mu$ . Note that if  $j + k < m$  then  $P_*^{0,j}\mu = P_*^{0,j}(x^k \Phi_*((P_*^{0,m-1}\mu)^\mathbb{Z}))$ . Indeed, let  $T$  be any subgroup of  $\mathcal{X}^{[0,j]}$ . On the one hand the set  $\{T\} \in S(\mathcal{X}^{[0,j]})$  the left hand side is equal to  $\mu(\{H < \mathcal{A}_n \mid H \cap \mathcal{X}^{[0,j]} = T\})$ , and the right hand side is equal to the measure  $\Phi_*((P_*^{0,m-1}\mu)^\mathbb{Z})$  of the set  $\{H < \mathcal{A}_n \mid H \cap \mathcal{X}^{[k,j+k]} = T\}$ . If  $[k, j+k] \subset [0, m-1]$  then it is further equal to the measure  $P_*^{0,m-1}\mu$  of the set  $\{H < \mathcal{X}^{[0,m-1]} \mid H \cap \mathcal{X}^{[k,j+k]} = T\}$  and hence to the measure  $\mu$  of the set

$$\{H < \mathcal{A}_n \mid (H \cap \mathcal{X}^{[0,m-1]}) \cap \mathcal{X}^{[k,j+k]} = T\},$$

and by shift invariance of  $\mu$  we have the equality. Therefore,  $\|P_*^{0,j}\mu_m - P_*^{0,j}\mu\|_1 \leq 2j/m$  which implies  $\lim_{m \rightarrow \infty} \mu_m = \mu$ . □

*Remark 1.* Denote by  $e_t$  the element  $(0, \dots, 1, \dots, 0) \in (\mathbb{Z}/p\mathbb{Z})^n$  with 1 on the  $t$ -th place, and by  $e_{t,k}$  the element  $(\dots, 0, e_t, 0, \dots) \in \mathcal{A}_n$  with  $e_t$  on the  $k$ -th place. Let also  $Y = \{1, \dots, n\}$ . There is a continuous inclusion  $q : Y^\mathbb{Z} \rightarrow S(\mathcal{A}_n)$  with a sequence  $(\varepsilon_k)_{k \in \mathbb{Z}}$  mapped to the

subgroup generated by  $e_{\varepsilon_k, k}$ , and it is easy to note that it commutes with shift. Hence it defines the inclusion map  $M_x(Y^{\mathbb{Z}}) \rightarrow M_x(\mathcal{A}_n)$ , where  $M_x(Y^{\mathbb{Z}})$  denotes the space of all shift invariant measures on  $Y^{\mathbb{Z}}$ . It is well-known that  $M_x(Y^{\mathbb{Z}})$  is Poulsen, thus this construction defines a Poulsen subsimplex in  $M_x(\mathcal{A}_n)$ .

Moreover, it is easy to see that  $q$  intertwines  $P^{i,j}$  with projection  $Y^{\mathbb{Z}} \rightarrow Y^{[i,j]}$  and the map  $\Phi$  with the natural identification of sets  $(Y^m)^{\mathbb{Z}}$  and  $Y^{\mathbb{Z}}$ . Thus the proof above, restricted to the image of  $q$ , gives another proof of the fact that  $M_x(Y^{\mathbb{Z}})$  is Poulsen.

*Remark 2.* If instead of  $\mathcal{A}_n$  we consider the compact group  $\hat{\mathcal{A}}_n = \prod_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})^n$ , and the set of its closed subgroups  $S(\hat{\mathcal{A}}_n)$ , then by replacing in the above proof the map  $\Phi$  by

$$\hat{\Phi}(\dots, H_{-1}, H_0, H_1, \dots) = \dots \times H_{-1} \times H_0 \times H_1 \times \dots$$

we have proof that the simplex  $M_x(\hat{\mathcal{A}}_n)$  of shift invariant measures on  $S(\hat{\mathcal{A}}_n)$  is Poulsen.

## References

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